

Thermodynamic bounds on equilibrium fluctuations of a global or local order parameter

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Abstract –We analyze thermodynamic bounds on equilibrium fluctuations of an order parameter, which are analogous to relations, which have been derived recently in the context of non-equilibrium fluctuations of currents. We discuss the case of *global* fluctuations when the order parameter is measured in the full system of interest, and *local* fluctuations, when the order parameter is evaluated only in a sub-part of the system. Using isometric fluctuation theorems, we derive thermodynamic bounds on the fluctuations of the order parameter in both cases. These bounds could be used to infer the value of symmetry breaking field or the relative size of the observed sub-system to the full system from *local* fluctuations.

Recently, a set of thermodynamic bounds have been obtained, which have a linear-response form and express a trade-off between the variance of current fluctuations and the rate of entropy production [1,2]. These relations contribute to the field of statistical kinetics and could represent important trade-offs in the design of living systems. Following this work, these uncertainty bounds have been derived rigorously from large deviation theory [3]. Specific bounds on current fluctuations have also been obtained separately for the symmetric exclusion process and for diffusive systems [4].

Dissipative systems break the time-reversal symmetry; but the formalism of large deviation theory is general and is also applicable to equilibrium fluctuations [5,6]. For equilibrium fluctuations, other forms of symmetry breaking not related to time are known. For instance, an ensemble of N Ising spins in a magnetic field is a classic illustration of an equilibrium system with discrete symmetry breaking. In discussing this pedagogical example [7], N. Goldenfeld derived a simple relation for the ratio of the probability to observe a magnetization \mathbf{M}_N , $P_{\mathbf{B}}(\mathbf{M}_N)$ with the probability to observe instead $-\mathbf{M}_N$:

$$P_{\mathbf{B}}(\mathbf{M}_N) = P_{\mathbf{B}}(-\mathbf{M}_N) e^{2\beta \mathbf{B} \cdot \mathbf{M}_N}. \quad (1)$$

The similarity of Eq. (1) with the Gallavotti-Cohen fluctuation theorem has been briefly noticed in [8] and only extensively studied in [9]. Inspired by these works and

by the discovery of fluctuation relations combining spatial and time-reversal symmetries called isometric fluctuation relations [10,11], one of us derived an extension of Eq. (1) for general symmetries described by group theory [12], which was then illustrated on a number of classic models of statistical physics [13].

In this paper, we derive analogs of the thermodynamic uncertainty bounds for equilibrium systems with symmetry breaking. Using Eq. (1), we find under some restrictive conditions to be detailed below, the following inequality for the variance of \mathbf{M}_N :

$$\frac{\text{Var}(M_N)}{\langle M_N \rangle} \leq \frac{k_B T}{B}, \quad (2)$$

where we denote the projection of \mathbf{M}_N along \mathbf{B} as M_N . In terms of the magnetization density $\mathbf{m} = \mathbf{M}_N/N$, this relation is equivalent to $\text{Var}(m)/\langle m \rangle \leq k_B T / BN$.

We will also prove a different inequality, which instead holds more generally for any system of classical spins (Ising or Heisenberg) and arbitrary couplings and reads:

$$\frac{\langle M_N^2 \rangle}{\langle M_N \rangle} \geq \frac{k_B T}{B}. \quad (3)$$

A remarkable feature of the inequalities of Eqs. (2)-(3) is that they hold beyond the linear response regime of small B , but become saturated when $B \rightarrow 0$ [7].

The outline of this paper is as follows: after providing an equivalent form of Eq. (2), we consider some simple cases with one or two spins, then we prove Eq. (3) for an ensemble of classical Heisenberg spins. The rest of the paper investigates the validity of Eq. (2) for large systems either using a *global* or a *local* order parameter.

For a finite number of spins, the magnetic susceptibility $\chi_N = d\langle M_N \rangle / dB$ satisfies the fluctuation-response relation $\chi_N = \beta \text{Var}(M_N)$ for any finite value of the magnetic field [14]. Thus, the inequality of Eq. (2) is equivalent to

$$\chi_N \leq \frac{\langle M_N \rangle}{B}. \quad (4)$$

It is reasonable that such a relation should hold independently of the temperature because it holds at least near $B = 0$ (where the inequality is saturated) and near $B \rightarrow \infty$. Indeed, in the latter case, the susceptibility vanishes due to the saturation of the magnetization, and the right hand side of Eq. (4) also vanishes because the average magnetization is bounded and $B \rightarrow \infty$. Whether the inequality should hold also in the intermediate range of values of B is the real question.

In order to investigate this, let us consider a simple case namely that of a single Ising spin $\sigma = \pm 1$ at temperature T in a magnetic field $B > 0$. Using the canonical distribution: $p(\sigma) = e^{\beta B \sigma} / Z$, with Z the partition function, it is straightforward to show that $\langle \sigma^2 \rangle = 1$ and $\langle \sigma \rangle = \tanh(\beta B)$. Our inequalities Eq. (3) and Eq. (2) are indeed verified since :

$$\frac{\langle \sigma^2 \rangle}{\langle \sigma \rangle} = \coth(\beta B) \geq \frac{k_B T}{B}, \quad (5)$$

and

$$\frac{\langle \sigma^2 \rangle - \langle \sigma \rangle^2}{\langle \sigma \rangle} = \frac{1 - \tanh^2(\beta B)}{\tanh(\beta B)} \leq \frac{k_B T}{B}. \quad (6)$$

Now, let us consider two such Ising spins σ_1 and σ_2 interacting with a coupling constant J again at temperature T and in a magnetic field $B > 0$. Naturally, we are interested in the fluctuations of the total magnetization, $M_2 = \sigma_1 + \sigma_2$. A straightforward calculation gives

$$\frac{\langle M_2^2 \rangle}{\langle M_2 \rangle} = 2 \coth(2\beta B) \geq \frac{k_B T}{B}, \quad (7)$$

which confirms Eq. (3) for $N = 2$. Now,

$$\frac{\text{Var}(M_2)}{\langle M_2 \rangle} = \frac{2(\exp(2\beta J) + \cosh(2\beta B))}{\sinh(2\beta B)(\exp(2\beta J) \cosh(2\beta B) + 1)}. \quad (8)$$

By maximizing the function on the right hand side of this equation over J with $J \geq 0$, one finds that the maximum is reached for $J = 0$. Therefore, $\text{Var}(M_2) / \langle M_2 \rangle \leq 2 / \sinh(2\beta B) \leq k_B T / B$, which confirms Eq. (2) for $N = 2$ in that case. However, note that if we allow antiferromagnetic couplings ($J < 0$), Eq. (2) can be violated in some range of values of B .

In order to understand this point, we plot in figure 1, the magnetic susceptibility χ_N and the quantity $\chi_N - \langle M_N \rangle / B$ for two spins with ferromagnetic or antiferromagnetic coupling. In the case of two spins with ferromagnetic coupling, χ_N is a monotonously decreasing function of the magnetic field B on an interval of the form $[0, \infty[$, and Eq. (4) holds. In contrast, for antiferromagnetic coupling, χ_N is non-monotonous and Eq. (4) is violated in a certain range of magnetic field. We shall come back to this interesting observation later.

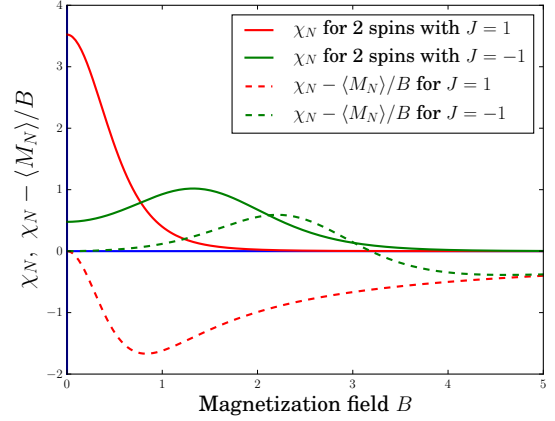


Fig. 1: Magnetic susceptibility χ_N (resp. $\chi_N - \langle M_N \rangle / B$) as a function of the magnetic field B for two spins with ferromagnetic ($J = 1$) as red solid line (resp. dashed) or with antiferromagnetic interaction ($J = -1$) as green solid line (resp. dashed).

We are now in position to generalize these results further. Let us consider an arbitrary ensemble of N classical spins $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_i\}_{i=1}^N$ taking discrete or continuous values such that $\boldsymbol{\sigma}_i \in \mathbb{R}^d$ and $\|\boldsymbol{\sigma}_i\| = 1$ [13]. The Hamiltonian of the system is assumed to be of the form

$$H_N(\boldsymbol{\sigma}; \mathbf{B}) = H_N(\boldsymbol{\sigma}; \mathbf{0}) - \mathbf{B} \cdot \mathbf{M}_N(\boldsymbol{\sigma}). \quad (9)$$

By a simple calculation (see appendix A for details), one obtains

$$\frac{\langle M_N^2 \rangle}{\langle M_N \rangle} = \frac{\int_{M_N > 0} d\mathbf{M}_N M_N^2 \cosh(\beta \mathbf{B} \cdot \mathbf{M}_N) P_0(\mathbf{M}_N)}{\int_{M_N > 0} d\mathbf{M}_N M_N \sinh(\beta \mathbf{B} \cdot \mathbf{M}_N) P_0(\mathbf{M}_N)}. \quad (10)$$

Now, we use the inequality $\tanh(x) \leq x$ for $x \geq 0$, which is equivalent to $\sinh(x) \leq x \cosh(x)$. By reporting the latter inequality into the denominator, one obtains

$$\frac{\langle M_N^2 \rangle}{\langle M_N \rangle} \geq \frac{\int_{M_N > 0} d\mathbf{M}_N M_N^2 \cosh(\beta B M_N) P_0(\mathbf{M}_N)}{\int_{M_N > 0} d\mathbf{M}_N \beta B M_N^2 \cosh(\beta B M_N) P_0(\mathbf{M}_N)}. \quad (11)$$

After simplifying the right hand side, we obtain Eq. (3) which is thus proven for any ensemble of classical spins with arbitrary couplings, as long as the system's Hamiltonian is given by Eq. (33).

This simple derivation does not work for Eq. (2), which is unfortunate because Eq. (2) is more informative than Eq. (3) - specially considering the large B limit - and is a closer analog to the nonequilibrium uncertainty relation [1–3]. For this reason, we focus below on Eq. (2).

Let us consider a large number of spins N , so that we can use the large deviation function [5, 6]:

$$P_{\mathbf{B}}(\mathbf{M}_N) \simeq e^{-N\Phi_{\mathbf{B}}(\mathbf{m})}. \quad (12)$$

Let us introduce the function

$$\Phi_{LR}(\mathbf{m}) = \beta \frac{(\mathbf{m} - \mathbf{m}^*)^2 \mathbf{B} \cdot \mathbf{m}^*}{2(m^*)^2}, \quad (13)$$

where \mathbf{m}^* is the most probable value of the magnetization which is such that $\Phi_{\mathbf{B}}(\mathbf{m}^*) = \Phi'_{\mathbf{B}}(\mathbf{m}^*) = 0$. This index LR in Φ_{LR} indicates that this is a linear response regime with respect to \mathbf{B} [3]. Using the fluctuation theorem of Eq. (1), it is easy to verify that $\Phi(\mathbf{m})$ and $\Phi_{LR}(\mathbf{m})$ take the same value and their derivatives are equal at the two symmetrically placed points $\pm \mathbf{m}^*$. To illustrate this geometrically, the two functions $\Phi_{\mathbf{B}}(\mathbf{m})$ and $\Phi_{LR}(\mathbf{m})$ are shown in fig. 2, for the particular case of the mean-field Curie-Weiss model in the ferromagnetic phase. In this case, note the concavity of $\Phi_{\mathbf{B}}(\mathbf{m})$ in the coexistence region $-m^* \leq m \leq m^*$. In contrast, this region becomes flat for the 2D Ising model in the limit $B \rightarrow 0$ [6].

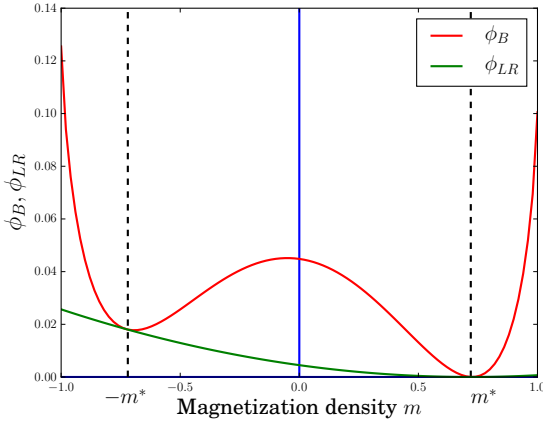


Fig. 2: Large deviation function $\Phi_{\mathbf{B}}(\mathbf{m})$ and its linear-response approximation $\Phi_{LR}(\mathbf{m})$ for the *global* fluctuations of the magnetization density \mathbf{m} in the Curie-Weiss model for a temperature $T = 0.8$ (in the ferromagnetic phase) and a magnetic field $B = 0.01$. The two functions are tangent at the points $\pm m^*$ the locations of which are shown by the two vertical dashed lines.

Now the variance of the order parameter is $\Phi''_{\mathbf{B}}(\mathbf{m}^*) = 1/(N\text{Var}(\mathbf{m}))$, and $\langle m \rangle = m^*$, because m^* is unique for large enough N . Thus, Eq. (2) holds if

$$\Phi''_{\mathbf{B}}(\mathbf{m}^*) \geq \frac{\beta B}{m^*}, \quad (14)$$

where the prime denotes the gradient component in the direction of \mathbf{B} . Now, since $\Phi_{\mathbf{B}}(\mathbf{m}) = \Phi_0(\mathbf{m}) - \Phi_0(\mathbf{m}^*) - \beta \mathbf{B} \cdot (\mathbf{m} - \mathbf{m}^*)$ [13], one has $\Phi'_{\mathbf{B}}(\mathbf{m}^*) = \Phi'_0(\mathbf{m}^*) - \beta \mathbf{B} = 0$ and $\Phi''_{\mathbf{B}}(\mathbf{m}) = \Phi''_0(\mathbf{m})$. Thus, the inequality Eq. (14) is implied by the positiveness of the function

$$h(\mathbf{m}) = \Phi''_0(\mathbf{m}) - \Phi'_0(\mathbf{m})/m, \quad (15)$$

which can not depend on the value of the magnetic field B . Further, $\Phi'_0(\mathbf{m}) = -\Phi'_0(-\mathbf{m})$ by symmetry and $\Phi'_0(\mathbf{0}) = 0$. Now, when $\Phi'_0(\mathbf{m})$ is convex for $m > 0$ and concave for $m < 0$, $h(\mathbf{m})$ is positive.

In other words, we must have basically $\Phi'''_0(\mathbf{m}) > 0$ for $m > 0$. Now, it is possible to relate this condition to the one found in our earlier study of the two spins. Indeed, since $\Phi'_0(\mathbf{m}^*) = \beta \mathbf{B}$, by taking a derivative with respect to B , one obtains $\chi_N = \beta N / \Phi''_0(\mathbf{m}^*)$. By taking a further derivative with respect to B , one finds $d\chi_N/dB = -\beta \chi_N \Phi'''_0(\mathbf{m}^*) / [\Phi''_0(\mathbf{m}^*)]^2$. Further, $\chi_N \geq 0$ by the fluctuation-response relation. Therefore, the condition $\Phi'''_0(\mathbf{m}) > 0$ is equivalent to the condition that the susceptibility be a monotonously decreasing function of B on the interval $B > 0$, which is the condition found earlier in our study of the two spins case. To summarize, the same condition must be met for Eq. (14) and therefore Eq. (2) to hold, both at the level of two spins or with a large number of them.

As an illustration, we can consider the Curie-Weiss model with Ising spins. The large deviation function of that model is [9]:

$$\Phi_{\mathbf{B}}(m) = I(m) - \frac{1}{2}\beta J m^2 - \beta B m - \beta f(B). \quad (16)$$

where $f(B)$ is the Helmholtz free energy per spin and $I(m)$ is the classic entropy function :

$$I(m) = \frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2}. \quad (17)$$

The most probable value of the magnetization, $m^*(B)$ given the magnetic field B , satisfies the relation $\Phi'_{\mathbf{B}}(m^*) = 0$, which leads to the well-known self-consistent equation $m^* = \tanh(\beta J m^* + \beta B)$. The function $\Phi_{\mathbf{B}}(m)$ is shown in fig. 2 in the ferromagnetic phase. Note that the region $[-m^*, m^*]$ defines the coexistence region, where $\Phi_{\mathbf{B}}(m)$ is concave.

The function $h(m)$ introduced in Eq. (15) is

$$h(m) = \frac{1}{1-m^2} - \frac{\tanh^{-1}(m)}{m}. \quad (18)$$

Since it is of the form $f'(m) - f(m)/m$, with f is convex for $m \geq 0$ and concave for $m \leq 0$, it follows that $h(m)$ is indeed positive. From the positiveness of $h(m)$, the inequality of Eq. (14) holds, which then implies the bound for the fluctuations of the *global* order parameter given by Eq. (2).

For the case of the 2D Ising model, we resort to numerical simulations since we are not able to check directly this

condition on the function $h(m)$. The results are shown in fig. 3. In order to test this, the difference between the left hand side and the right hand side in Eq. (2) is plotted as a function of B , so that all the points should be below the red line $y = 0$ according to the inequality. The errorbars have been estimated using the method of Ref. [15]. These errorbars increase rapidly as $B \rightarrow 0$ in a system size dependent manner due to the singularity in the derivative of the free energy $F(\mathbf{B})$ at $B = 0$. We confirm that the bound holds for the paramagnetic phase (upper figure) as well as for the ferromagnetic phase (lower figure). It is more tight in the former case than in the latter, since the fluctuations are Gaussian in the former case. This analyt-

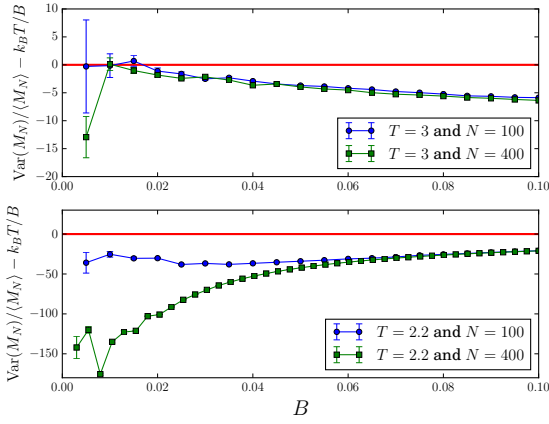


Fig. 3: Difference between the left and right hand side of Eq. (2) for the 2D-Ising model as a function of the magnetic field B , for two sizes $N = 100$ and $N = 400$. The temperature is $T = 3.0$ (paramagnetic phase) in the upper figure and $T = 2.2$ (ferromagnetic phase) in the lower figure.

ical and numerical study of the thermodynamic bounds of Eqs. (2)-(3) represents our first main result.

We now investigate how such bounds are modified when we do not have access to the *global* order parameter, but rather to a coarse-grained or *local* one. Since the bounds are related to the fluctuation theorem, we need to generalize Eq. (1) for such a case. A similar situation arises out of equilibrium due to coarse-graining [16–21]. Understanding how to extract relevant information in such cases is rather pertinent experimentally even at equilibrium since *local* measurements are often the only choice, in the frequent case that the system is just too big to be analyzed globally.

In order to study a *local* version of Eq. (1), we consider a subset of the N spins containing $n < N$ spins only, $\Lambda = \{\sigma_i\}_{i=1}^n$, with magnetization $\mathbf{M}_n(\sigma) = \sum_{i=1}^n \sigma_i$ as shown in fig. 4. The remaining spins $\bar{\Lambda} = \{\sigma_i\}_{i=n+1}^N$ play the role of an “environment” for the spins of Λ . This environment has a magnetization $\bar{\mathbf{M}}_n(\sigma) = \sum_{i=n+1}^N \sigma_i$, so that $\mathbf{M}_N = \mathbf{M}_n + \bar{\mathbf{M}}_n$. The local equivalent of Eq. (1) is:

$$P_{\mathbf{B}}(\mathbf{M}_n) = P_{\mathbf{B}}(-\mathbf{M}_n) e^{\beta(2\mathbf{B} \cdot \mathbf{M}_n + \Gamma_{\mathbf{B}}(\mathbf{M}_n))}. \quad (19)$$

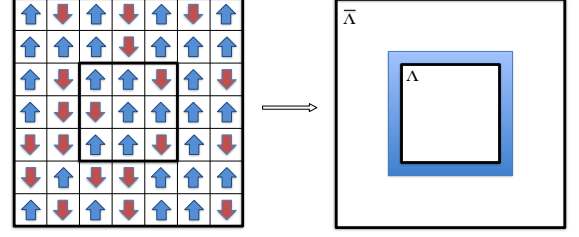


Fig. 4: Sketch of the magnetic system with a central square region Λ representing the observation window. Spins in the complementary region $\bar{\Lambda}$ belonging to the blue area are strongly coupled to that of Λ .

We have introduced the function

$$\begin{aligned} \Gamma_{\mathbf{B}}(\mathbf{M}_n) &= k_B T \ln \langle e^{-2\beta \mathbf{B} \cdot \bar{\mathbf{M}}_n} | -\mathbf{M}_n \rangle_{\mathbf{B}}, \\ &= k_B T \ln \int e^{-2\beta \mathbf{B} \cdot \bar{\mathbf{M}}_n} \text{Proba}(\bar{\mathbf{M}}_n | -\mathbf{M}_n) d^3 \bar{\mathbf{M}}_n, \end{aligned} \quad (20)$$

where $\text{Proba}(\bar{\mathbf{M}}_n | -\mathbf{M}_n)$ denotes the conditional probability of $\bar{\mathbf{M}}_n$ given a magnetization $-\mathbf{M}_n$ for the sub-part. Thus, $\Gamma_{\mathbf{B}}(\mathbf{M}_n)$ is a correction factor which quantifies the failure of Eq. (1) due to the reduction of available information in the fluctuations. By construction, this factor must be an odd function of \mathbf{M}_n , i. e.: $\Gamma_{\mathbf{B}}(-\mathbf{M}_n) = -\Gamma_{\mathbf{B}}(\mathbf{M}_n)$.

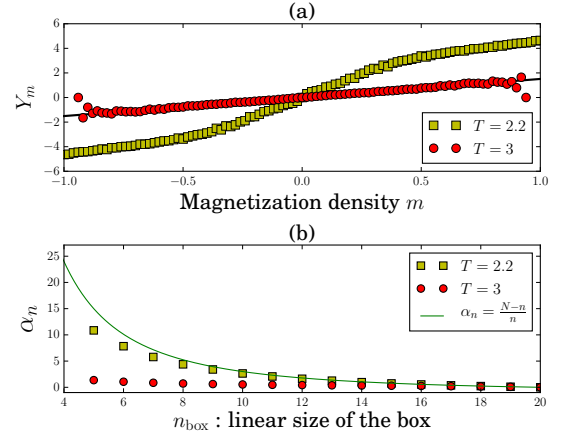


Fig. 5: (a) Asymmetry function Y_m versus magnetization density m for the 2D Ising model in a magnetic field, at a temperature $T = 3$ (above T_c) or $T = 2.2$ (below T_c). The order parameter is evaluated in a sub-part of $n = 100$ spins among a total of $N = 400$ spins, and the magnetic field is $B = 0.01$. The critical temperature is $T_c \simeq 2.38$ for this system size. (b) Dependence of α_n versus the size of the sub-part (symbols) for the same two temperatures. The solid line shows the dependence which is expected as the critical point is approached.

In the case that all the spins of the sub-part and the rest are independent, there are no correlations between Λ

and $\bar{\Lambda}$, which means $\text{Proba}(\bar{\mathbf{M}}_n | -\mathbf{M}_n) = \text{Proba}(\bar{\mathbf{M}}_n)$. Using a Jarzynski like relation immediately deduced from Eq. (1) for the complementary part, one has $\Gamma_{\mathbf{B}}(\mathbf{M}_n) = \Omega_{\mathbf{B}}(\mathbf{M}_n) = 0$. Therefore, the breaking of the Fluctuation relations Eq. (1) arises entirely from the correlations between the domains Λ and $\bar{\Lambda}$.

In order to simplify this problem, we further split the “environment”, namely $\bar{\Lambda}$, into a subset of strongly correlated spins (the blue area in figure 4), and the rest of the spins, which are less correlated [18]. This can be written as:

$$\bar{\mathbf{M}}_n = \alpha_n \mathbf{M}_n + \xi_n, \quad (21)$$

where we require ξ_n to be uncorrelated with \mathbf{M}_n . In particular, this form should hold above T_c , where the correlation length is small and all the spins of $\bar{\Lambda}$ are uncorrelated with that of Λ except for those at the interface between both domains.

Since ξ_n is uncorrelated with \mathbf{M}_n , α_n equals the normalized co-variance between \mathbf{M}_n and $\bar{\mathbf{M}}_n$:

$$\alpha_n = \frac{\langle \mathbf{M}_n \cdot \bar{\mathbf{M}}_n \rangle - \langle \mathbf{M}_n \rangle \cdot \langle \bar{\mathbf{M}}_n \rangle}{\langle \mathbf{M}_n^2 \rangle - \langle \mathbf{M}_n \rangle^2}. \quad (22)$$

Then, using Eq. (20) and Eq. (21), one finds a linear correction $\Gamma_{\mathbf{B}}(\mathbf{M}_n) = 2\alpha_n \mathbf{M}_n \cdot \mathbf{B}$. The asymmetry function Y_m defined by

$$Y_m = \frac{1}{2\beta B n} \ln \frac{P_{\mathbf{B}}(m)}{P_{\mathbf{B}}(-m)}, \quad (23)$$

is a straight line of slope one for the *global* order parameter due to Eq. (1) but becomes a straight line of slope $1 + \alpha_n$ for the *local* order parameter. When α_n does not depend on the magnetization, the change of slope can be described by the inverse effective temperature $\beta_{eff} = \beta(1 + \alpha_n)$ or by an effective magnetic field, similarly to the nonequilibrium case [16]. Since the magnetization of $\bar{\Lambda}$ acts like a field for Λ enhancing its magnetization, $\alpha_n \geq 0$ and this effective temperature is smaller than T . A straight asymmetry function with a slope larger than one is indeed found in figure 5a, when analyzing the fluctuations in a box of $n = 100$ spins among a total of $N = 400$ spins at the temperature $T = 3$.

As $T \rightarrow T_c^+$, the correlation length increases until it becomes of the order of the size of the full system. Then, the contribution of ξ_n in Eq. (21) should vanish on average, and the average magnetization density is $\langle \mathbf{m} \rangle = \langle \mathbf{M}_n \rangle / n = \langle \bar{\mathbf{M}}_n \rangle / (N - n)$, which implies $\alpha_n \simeq (N - n) / n$. Away from the critical point, α_n also scales as $1/n$ but the prefactor does not have such a simple form. We have checked numerically that indeed $\alpha_n \simeq (N - n) / n$ near the critical point as shown in fig. 5b. Using such a determination of α_n , one could infer the relative size of the observation window to the size of the large system. In contrast, below the critical point, the asymmetry function of the *local* order parameter, has a sigmoidal shape as shown in fig. 5 when the temperature is $T = 2.2$. A

similar shape is found in the case of the mean-field Curie Weiss model which is completely solvable analytically (see appendix B for details of the derivation of the correction factor $\Gamma_{\mathbf{B}}(\mathbf{M}_n)$ for this case).

Let us now finally go back to our initial topic of thermodynamic bounds of the type of Eq. (2) but now for *local* fluctuations of the order parameter. The relevant large deviation function is defined as

$$P_{\mathbf{B}}(\mathbf{M}_n) \simeq e^{-n\phi_{\mathbf{B}}(\mathbf{m})}, \quad (24)$$

for n sufficiently large. Below, we use the same notation for the magnetization density $\mathbf{m} = \mathbf{M}_n / n$. In view of the modified fluctuation theorem of Eq. (19), the approximation

$$\phi_{LR}(\mathbf{m}) = \beta B Y_{m^*} \frac{(\mathbf{m} - \mathbf{m}^*)^2}{2(m^*)^2}, \quad (25)$$

is correct by construction close to $\mathbf{m} = \mathbf{m}^*$ and has the expected value at $\mathbf{m} = -\mathbf{m}^*$ but unlike Φ_{LR} may not have the correct tangent at this point.

To see precisely when this property holds, we start with an equivalent form of Eq. (19), namely:

$$\phi_{\mathbf{B}}(\mathbf{m}) - \phi_{\mathbf{B}}(-\mathbf{m}) = -2\beta \mathbf{B} \cdot \mathbf{m} - \frac{\beta \Gamma(n\mathbf{m})}{n}, \quad (26)$$

with $\Gamma(n\mathbf{m})$ related to the asymmetry function Y_m by

$$Y_m = m + \frac{\Gamma(n\mathbf{m})}{2Bn}. \quad (27)$$

Using Eq. (26) and the property $\phi'_{\mathbf{B}}(\mathbf{m}^*) = 0$, one deduces that

$$\phi'_B(-\mathbf{m}^*) = -2\beta B - \beta \Gamma'(n\mathbf{m}^*), \quad (28)$$

while from Eq. (25), one obtains

$$\phi'_{LR}(-\mathbf{m}^*) = -2\beta B - \beta \frac{\Gamma(n\mathbf{m}^*)}{nm^*}. \quad (29)$$

Therefore, we see that $\phi'_B(-\mathbf{m}^*) = \phi'_{LR}(-\mathbf{m}^*)$ if and only if

$$\Gamma'(n\mathbf{m}^*) = \frac{\Gamma(n\mathbf{m}^*)}{nm^*}. \quad (30)$$

From this and given that $\Gamma(0) = 0$, this condition is satisfied whenever (i) $m^* \rightarrow 0$, which is for instance the case when $\mathbf{B} \rightarrow \mathbf{0}$ and $T > T_c$, (ii) the size of the sub-part goes to zero $n \rightarrow 0$, or more generally (iii) the asymmetry function Y_m is a linear function of m of the form $Y_m = (1 + \alpha_n)m$, where α_n is the coefficient introduced earlier. When one of these conditions hold, the function ϕ_{LR} approximates $\phi_{\mathbf{B}}$ for all values of the magnetization, because both functions are tangent at $\mathbf{m} = \pm m^*$.

In such a case, the previous derivation of the thermodynamic bound applies directly in terms of the effective field $B_{eff} = (1 + \alpha_n)B$, so that the generalization of Eq. (2) is:

$$\frac{\text{Var}(M_n)}{\langle M_n \rangle} \leq \frac{k_B T}{B_{eff}}, \quad (31)$$

or $\text{Var}(m)/\langle m \rangle \leq k_B T / n B_{\text{eff}}$ for the magnetization density.

As a particular case, the result holds for the Curie-Weiss model. Indeed, after a straightforward calculation, the large deviation for the fluctuations of the *local* order parameter defined in Eq. (24) reads [13],

$$\begin{aligned} \phi_{\mathbf{B}}(m) &= I(m) - \beta B m - \frac{\beta J}{2} m^2 \frac{n}{N} - \frac{N \beta f(B)}{n} \\ &+ \beta f(B + J n m / N) \left(\frac{N}{n} - 1 \right). \end{aligned} \quad (32)$$

This expression allows to compare the large deviation function $\phi_{\mathbf{B}}(m)$ and its linear-response approximation $\phi_{LR}(m)$ for various size ratios of the sub-part to the full system. For general values of the size ratio and within the ferromagnetic phase, we have checked that ϕ_{LR} is indeed not tangent at the point $m = -m^*$ although both functions $\phi_{LR}(m)$ and $\phi_{\mathbf{B}}(m)$ take the same value there. When considering smaller size ratios of the sub-part to the full system or within the paramagnetic phase, the two curves become tangent at $m = -m^*$. In such conditions, the bound on the fluctuations of the *local* fluctuations, Eq. (31) holds.

For the case of the 2D Ising model, we use again numerical simulations. The results are shown in fig. 6, where the difference between the left and right hand side of Eq. (31) is shown, in the paramagnetic phase with the appropriate expression of B_{eff} for two choices for the size of the sub-part $n = 25$ or $n = 100$. This verification confirms the bound for *local* fluctuations for this model, which represents our second main result.

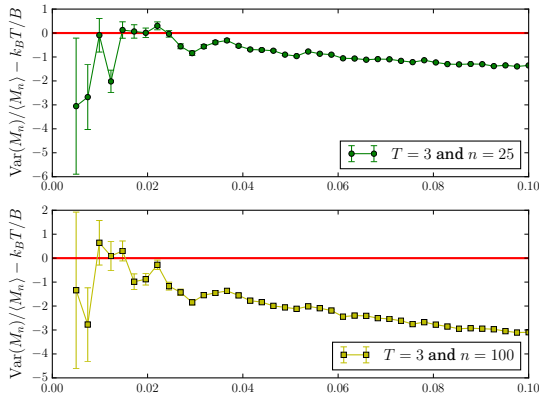


Fig. 6: Difference between the left and right hand side of Eq. (31) as a function of B confirming the bound for *local* fluctuations for the case of the 2D-Ising model at a temperature $T = 3$ (in the paramagnetic phase) for two sizes of the sub-part $n = 100$ and $n = 25$. The values of effective magnetic field have been evaluated using the α_n of fig. 5.

To conclude, we have derived thermodynamic bounds on equilibrium fluctuations of *global* and *local* order parameters. The bound for the fluctuations of a *global* order

parameter is analogous to the one derived recently out of equilibrium [1–3]. In this formal analogy, the average entropy production must be replaced by the magnetic field. This is expected since out of equilibrium, the entropy production quantifies the degree of breaking of time-reversal symmetry, while in equilibrium, the magnetic field is responsible for the breaking of the spatial discrete symmetry.

The two thermodynamic bounds contain the following trade-offs: Out of equilibrium, the bound imposes that reducing current fluctuations costs a minimal dissipation [1–3]; in equilibrium, Eq. (2) imposes that reducing order parameter fluctuations can be achieved by increasing the magnetic field (which therefore costs some energy).

In these two relations, fluctuations are measured by their *variance*. If we choose instead to measure fluctuations by the average of the square of the order parameter, the picture which emerges from Eq. (3) is rather different: such a relation can describe situations where fluctuations diverge and order can be destroyed provided the average magnetization scales appropriately with B . For instance, near a critical point $\langle M_N \rangle_{T_c} \sim B^{1/\delta}$ [14], then Eq. (3) implies that $\langle M_N^2 \rangle_{T_c} \geq k_B T B^{1/\delta-1}$, which means that $\langle M_N^2 \rangle_{T_c}$ diverges at $B = 0$ if $\delta > 1$.

Both in and out of equilibrium, the bounds do not follow mainly from the fluctuation theorem, since additional properties are needed. In the present equilibrium case, we have seen an illustration of this with the example of the two spins. There, we found that Eq. (2) holds whenever the susceptibility is a monotonously decreasing function of the magnetic field. Considering instead a large ensemble of spins, we recovered the same condition, in the form of the positivity of third derivative of the large deviation function of the magnetization in the zero-field model. Our numerical study of the 2D Ising model confirms that the inequality holds for this model at any temperature and system size but requires ferromagnetic interactions.

Using the formalism of Fluctuation Theorems in the presence of hidden degrees of freedom developed for the non-equilibrium case [16], we have extended the uncertainty bounds to *local* order parameters. Such bounds are important because they can be tested experimentally more easily than their large deviation counterparts (whether at equilibrium or out of equilibrium). They could be used to infer the value of the symmetry breaking field, the relative size of the observation window with respect to the full system and possibly the nature of the interactions (ferromagnetic vs antiferromagnetic) using only fluctuations of the order parameter.

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Appendix A: A general inequality in the canonical ensemble

Here, we provide details for the derivations of Eqs. (3), (4) and (20) of the main text. Let us consider a system composed of N classical spins $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}_i\}_{i=1}^N$ taking discrete or continuous values such that $\boldsymbol{\sigma}_i \in \mathbb{R}^d$ and $\|\boldsymbol{\sigma}_i\| = 1$. The Hamiltonian of the system is assumed to be of the form

$$H_N(\boldsymbol{\sigma}; \mathbf{B}) = H_N(\boldsymbol{\sigma}; \mathbf{0}) - \mathbf{B} \cdot \mathbf{M}_N(\boldsymbol{\sigma}) \quad (33)$$

where \mathbf{B} is the external magnetic field and the order parameter is the magnetization

$$\mathbf{M}_N(\boldsymbol{\sigma}) = \sum_{i=1}^N \boldsymbol{\sigma}_i. \quad (34)$$

We suppose that the system is at equilibrium in the Gibbsian canonical distribution at the inverse temperature β , and we introduce

$$\mu_{\mathbf{B}}(\boldsymbol{\sigma}) = \frac{1}{Z_N(\mathbf{B})} e^{-\beta H_N(\boldsymbol{\sigma}; \mathbf{B})}, \quad (35)$$

where $Z_N(\mathbf{B}) = \sum_{\boldsymbol{\sigma}} e^{-\beta H_N(\boldsymbol{\sigma}; \mathbf{B})}$ is the classical partition function such that the distribution is normalized to unity: $\sum_{\boldsymbol{\sigma}} \mu_{\mathbf{B}}(\boldsymbol{\sigma}) = 1$.

Let us also define the probability density $P_{\mathbf{B}}(\mathbf{M})$ that the magnetization takes the value $\mathbf{M} = \mathbf{M}_N(\boldsymbol{\sigma})$ as

$$P_{\mathbf{B}}(\mathbf{M}) \equiv \langle \delta[\mathbf{M} - \mathbf{M}_N(\boldsymbol{\sigma})] \rangle_{\mathbf{B}} \quad (36)$$

where $\delta(\cdot)$ denotes the Dirac delta distribution and $\langle \cdot \rangle_{\mathbf{B}}$ the statistical average over Gibbs' canonical measure (35). The same distribution in the absence of the field, is denoted $P_{\mathbf{0}}(\mathbf{M})$, and is related to $P_{\mathbf{B}}(\mathbf{M})$ by

$$P_{\mathbf{B}}(\mathbf{M}) = \frac{Z_N(\mathbf{0})}{Z_N(\mathbf{B})} e^{\beta \mathbf{B} \cdot \mathbf{M}} P_{\mathbf{0}}(\mathbf{M}). \quad (37)$$

This probability density $P_{\mathbf{B}}(\mathbf{M})$ is a function of the vectorial magnetization $\mathbf{M} \in \mathbb{R}^d$ and it is normalized according to

$$\int d\mathbf{M} P_{\mathbf{B}}(\mathbf{M}) = 1. \quad (38)$$

We also define M_z to be the component of \mathbf{M} along the direction of \mathbf{B} , and B is the amplitude of \mathbf{B} , so that $\mathbf{B} \cdot \mathbf{M} = BM_z$.

Then, using Eq. (37)

$$\begin{aligned} \langle \mathbf{M}^2 \rangle &= \int d\mathbf{M} \mathbf{M} \mathbf{M}^2 P_{\mathbf{B}}(\mathbf{M}) \\ &= \int_{M_z > 0} d\mathbf{M} \mathbf{M} \mathbf{M}^2 2 \cosh(\beta \mathbf{B} \cdot \mathbf{M}) P_{\mathbf{0}}(\mathbf{M}) \frac{Z_N(\mathbf{0})}{Z_N(\mathbf{B})}. \end{aligned}$$

Similarly,

$$\begin{aligned} \langle M_z \rangle &= \int d\mathbf{M} M_z P_{\mathbf{B}}(\mathbf{M}) \\ &= \int_{M_z > 0} d\mathbf{M} M_z 2 \sinh(\beta \mathbf{B} \cdot \mathbf{M}) P_{\mathbf{0}}(\mathbf{M}) \frac{Z_N(\mathbf{0})}{Z_N(\mathbf{B})}. \end{aligned}$$

We are interested in the Fano factor like ratio

$$\frac{\langle \mathbf{M}^2 \rangle}{\langle M_z \rangle} = \frac{\int_{M_z > 0} d\mathbf{M} M^2 \cosh(\beta \mathbf{B} \cdot \mathbf{M}) P_0(\mathbf{M})}{\int_{M_z > 0} d\mathbf{M} M_z \sinh(\beta \mathbf{B} \cdot \mathbf{M}) P_0(\mathbf{M})}. \quad (39)$$

Now using $\mathbf{M}^2 \geq M_z^2$, we obtain

$$\frac{\langle \mathbf{M}^2 \rangle}{\langle M_z \rangle} \geq \frac{\langle M_z^2 \rangle}{\langle M_z \rangle} = \frac{\int_{M_z > 0} d\mathbf{M} M_z^2 \cosh(\beta \mathbf{B} \cdot \mathbf{M}) P_0(\mathbf{M})}{\int_{M_z > 0} d\mathbf{M} M_z \sinh(\beta \mathbf{B} \cdot \mathbf{M}) P_0(\mathbf{M})}. \quad (40)$$

Then, using $\tanh(x) \leq x$ for $x \geq 0$, which is equivalent to $\sinh(x) \leq \cosh(x)x$ in the denominator, we obtain

$$\frac{\langle M_z^2 \rangle}{\langle M_z \rangle} \geq \frac{\int_{M_z > 0} d\mathbf{M} M_z^2 \cosh(\beta B M_z) P_0(\mathbf{M})}{\int_{M_z > 0} d\mathbf{M} \beta B M_z^2 \cosh(\beta B M_z) P_0(\mathbf{M})}. \quad (41)$$

Therefore, we obtain our result, namely Eq. (3) of the main text (where M_z is simply denoted M_N):

$$\frac{\langle \mathbf{M}^2 \rangle}{\langle M_z \rangle} \geq \frac{\langle M_z^2 \rangle}{\langle M_z \rangle} \geq \frac{1}{\beta B}. \quad (42)$$

Note that this result holds generally for any magnetic system described by Eq. (33): there are no conditions on the number of spins or on the couplings between the spins.

The fluctuation-response relation given above Eq. (4) of the main text is obtained by taking the derivative of Eq. (39) with respect to the magnetic field B . This classic calculation leads to:

$$\frac{d\langle M_z \rangle}{dB} = \beta (\langle M_z^2 \rangle - \langle M_z \rangle^2) = \beta \text{Var} M_z. \quad (43)$$

This quantity is introduced in the main text above Eq. (4) as the magnetic susceptibility χ_N and together with Eq. (3), it leads to Eq. (4) of the main text.

Appendix B: Asymmetry function for the fluctuations of the local order parameter in the mean-field Curie Weiss model

Here, we provide derivation of the correction factor defined in Eq. (20) of the main text for the mean field Curie-Weiss model with Ising spins. The distribution of the *local* order parameter is

$$P_B(\mathbf{M}_n) = \frac{1}{Z_N(\mathbf{B})} \sum_{\sigma} e^{\frac{\beta J}{2N} \mathbf{M}_N^2 + \beta \mathbf{B} \cdot \mathbf{M}_N} \delta[\mathbf{M}_n - \mathbf{M}_n(\sigma)], \quad (44)$$

where the sum is taken over configurations of the full system. After separating the contribution of spins belonging to Λ and to $\bar{\Lambda}$, one obtains

$$P_B(\mathbf{M}_n) = \frac{Z_{\bar{\Lambda}}(\mathbf{B} + J\mathbf{M}_n/N)}{Z_N(\mathbf{B})} e^{\frac{\beta J}{2N} \mathbf{M}_n^2 + \beta \mathbf{B} \cdot \mathbf{M}_n} C_n(\mathbf{M}_n), \quad (45)$$

where $Z_{\bar{\Lambda}}(\mathbf{B} + J\mathbf{M}_n/N)$ is the partition function of an ensemble of $N - n$ spins of $\bar{\Lambda}$, subject to the effective magnetic field $\mathbf{B}_{eff} = \mathbf{B} - J\mathbf{M}_n/N$ and interacting with the effective coupling constant $J_{eff} = J(N - n)/N$. The function $C_n(\mathbf{M}_n)$ is the number of microstates with a given value of local magnetization \mathbf{M}_n . Since this function only depends on $|\mathbf{M}_n|$, one obtains the correction factor defined in Eq. (20) of the main text:

$$\Gamma_B(\mathbf{M}_n) = F_{\bar{\Lambda}}(\mathbf{B} - J\mathbf{M}_n/N) - F_{\bar{\Lambda}}(\mathbf{B} + J\mathbf{M}_n/N), \quad (46)$$

where $F_{\bar{\Lambda}}$ is the mean-field free energy associated with the partition function $Z_{\bar{\Lambda}}$.